SOLUTION SET IV

EXERCISE IV.1 : PERIODIC FUNCTIONS

Average value : $F_m = \frac{1}{T}\int_0^T f(t)dt$ Rms value : $F = \sqrt{\frac{1}{T}\int_0^T f^2(t)dt}$

Function 1 :

$$U_m = \frac{1}{T}\left[\int_0^{T/2} \hat{U}dt - \int_{T/2}^{T} \hat{U}dt\right] = \underline{0\ V} \qquad U = \sqrt{\frac{1}{T}\left[\int_0^{T/2} \hat{U}^2dt + \int_{T/2}^{T} \hat{U}^2dt\right]} = \hat{U} = \underline{20\ V}$$

$$U_{max} = \hat{U} = \underline{20\ V} \qquad U_{min} = \underline{-20\ V}$$

Function 2 :

$$U_m = \frac{1}{T}\int_0^{T/2} \hat{U}dt = \frac{1}{2}\hat{U} = \underline{5\ V} \qquad U = \sqrt{\frac{1}{T}\int_0^{T/2} \hat{U}^2dt} = \frac{1}{\sqrt{2}}\hat{U} = \underline{\frac{10}{\sqrt{2}}\ V}$$

$$U_{max} = \hat{U} = \underline{10\ V} \qquad U_{min} = \underline{0\ V}$$

Function 3 :

$$U_m = \frac{1}{T}\left[\int_0^{T/2} \hat{U}_+ dt - \int_{T/2}^{T} \hat{U}_- dt\right] = \frac{1}{T}\left[10\frac{T}{2} - 20\frac{T}{2}\right] = \underline{-5\ V}$$

$$U = \sqrt{\frac{1}{T}\left[\int_0^{T/2} \hat{U}_+{}^2 dt + \int_{T/2}^{T} \hat{U}_-{}^2 dt\right]} = \sqrt{\frac{1}{T}\left[10^2\frac{T}{2} + (-20)^2\frac{T}{2}\right]} = \sqrt{250} \approx \underline{15.8\ V}$$

$$U_{max} = \hat{U}_+ = \underline{10\ V} \qquad U_{min} = \hat{U}_- = \underline{-20\ V}$$

Function 4 :

$$I_m = \frac{1}{T}\int_0^T \hat{I}\sin(\omega t)dt = -\frac{1}{T\omega}\hat{I}\cos\left(\frac{2\pi}{T}t\right)\Big|_0^T = \underline{0A}$$

$$I = \sqrt{\frac{1}{T}\int_0^T \hat{I}^2\sin^2(\omega t)dt} = \sqrt{\frac{1}{T}\hat{I}^2\left[\frac{1}{2}t - \frac{1}{4\cdot\frac{2\pi}{T}}\cdot\sin(2\frac{2\pi}{T}t)\right]_0^T} = \frac{\hat{I}}{\sqrt{2}} = \underline{\sqrt{2}\ A}$$

$$I_{max} = \underline{2\ A} \qquad I_{min} = \underline{-2\ A}$$

Function 5 :

$$I_m = \frac{2}{T}\int_0^{T/2} \hat{I}\sin(\omega t)dt = -\frac{2}{T\frac{2\pi}{T}}\hat{I}\cos\left(\frac{2\pi}{T}t\right)\Big|_0^{T/2} = \frac{2}{\pi}\hat{I} = \underline{\frac{4}{\pi}A}$$

$$I = \sqrt{\frac{2}{T}\int_0^{T/2} \hat{I}^2\sin^2(\omega t)dt} = \sqrt{\frac{2}{T}\hat{I}^2\cdot\left[\frac{1}{2}t - \frac{1}{4\frac{2\pi}{T}}\sin\left(2\frac{2\pi}{T}t\right)\right]_0^{T/2}} = \frac{\hat{I}}{\sqrt{2}} = \underline{\sqrt{2}\ A}$$

$$I_{max} = \underline{2\ A} \qquad I_{min} = \underline{0\ A}$$

Function 6 :

$$I_m = \frac{1}{T}\int_0^{T/2} \hat{I}\sin(\omega t)dt = -\frac{1}{T\frac{2\pi}{T}}\hat{I}\cos\left(\frac{2\pi}{T}t\right)\Bigg|_0^{T/2} = \frac{1}{\pi}\hat{I} = \underline{\frac{2}{\pi}\,A}$$

$$I = \sqrt{\frac{1}{T}\int_0^{T/2}\hat{I}^2\sin^2(\omega t)dt} = \sqrt{\frac{1}{T}\hat{I}^2\cdot\left[\frac{1}{2}t - \frac{1}{4\frac{2\pi}{T}}\sin\left(2\frac{2\pi}{T}t\right)\right]_0^{T/2}} = \frac{\hat{I}}{2} = \underline{1\,A}$$

$$I_{max} = \underline{2\,A} \qquad I_{min} = \underline{0\,A}$$

EXERCISE IV.2 : EQUATIONS IN A AC CIRCUIT

A. Definition of voltages and currents :

The choice of the direction of the voltage drops as well as the designations of the voltages and currents is only one solution among others (convention).

B.

$$u_{R_1} = R_1 i_1$$

$$u_{R_2} = R_2 i_2$$

$$u_{C_1} = \frac{1}{C_1}\int_{t_0}^{t} i_2 dt + u_{C_1}(t_0)$$

$$u_{C_2} = \frac{1}{C_2}\int_{t_0}^{t} i dt + u_{C_2}(t_0)$$

$$u_{L_1} = L_1\frac{di_1}{dt}$$

$$u_{L_2} = L_2\frac{di}{dt}$$

$$i = i_1 + i_2$$

$$u = u_1 + u_2$$

$$u_1 = u_{R_1} + u_{L_1}$$

$$u_1 = u_{C_1} + u_{R_2}$$

$$u_2 = u_{C_2} + u_{L_2}$$

C.

$$\underline{U}_{R_1} = R_1\,\underline{I}_1$$

$$\underline{U}_{R_2} = R_2\,\underline{I}_2$$

$$\underline{U}_{C_1} = \frac{\underline{I}_2}{j\omega C_1}$$

$$\underline{U}_{C_2} = \frac{\underline{I}}{j\omega C_2}$$

$$\underline{U}_{L_1} = j\omega L_1\underline{I}_1$$

$$\underline{U}_{L_2} = j\omega L_2\underline{I}$$

$$\underline{I} = \underline{I}_1 + \underline{I}_2$$

$$\underline{U} = \underline{U}_1 + \underline{U}_2$$

$$\underline{U}_1 = \underline{U}_{R_1} + \underline{U}_{L_1}$$

$$\underline{U}_1 = \underline{U}_{C_1} + \underline{U}_{R_2}$$

$$\underline{U}_2 = \underline{U}_{C_2} + \underline{U}_{L_2}$$

D. There are 11 equations, 12 unknowns. It is therefore necessary to know a current or a voltage in addition to the value of the components of the circuit.

E. From $u_1 = 20\sqrt{2}\cos(\omega t)$ we get : $\underline{U}_1 = 20e^{j\,0^\circ}$

$$\underline{U}_1 = \underline{U}_{R_1} + \underline{U}_{L_1} = R_1\underline{I}_1 + j\omega L_1\underline{I}_1 = (R_1 + j\omega L_1)\,\underline{I}_1$$

$$R_1 + j\omega L_1 = 4 + j\,314{,}16\cdot 25\cdot 10^{-3} = 4 + j\,7{,}854 = \underline{8{,}814e^{j\,63{,}0^\circ}\,\Omega}$$

$$\underline{I}_1 = \frac{\underline{U}_1}{R_1 + j\omega L_1} = \frac{20e^{j\,0^\circ}}{8{,}814e^{j\,63{,}0^\circ}} = \underline{2{,}269e^{-j\,63{,}0^\circ}\,A} = \underline{(1{,}030 - j\,2{,}022)\,A}$$

$$\underline{U}_{R_1} = R_1\underline{I}_1 = 4\cdot 2{,}269e^{-j\,63{,}0^\circ} = \underline{9{,}076e^{-j\,63{,}0^\circ}\,V} = \underline{(4{,}12 - j\,8{,}087)\,V}$$

$$\underline{U}_{L_1} = j\omega L_1\underline{I}_1 = 7{,}854e^{j\,90{,}0^\circ}\cdot 2{,}269e^{-j\,63{,}0^\circ} = \underline{17{,}821e^{j\,27{,}0^\circ}\,V} = \underline{(15{,}88 + j\,8{,}09)\,V}$$

$$\underline{U}_1 = \underline{U}_{R_2} + \underline{U}_{C_1} = R_2\underline{I}_2 - j\frac{1}{\omega C_1}\underline{I}_2 = \left(R_2 - j\frac{1}{\omega C_1}\right)\underline{I}_2$$

$$R_2 - j\frac{1}{\omega C_1} = \underline{8 - j\,6{,}366\,\Omega} = \underline{10{,}224e^{-j\,38{,}5^\circ}\,\Omega}$$

$$\underline{I}_2 = \frac{\underline{U}_1}{R_2 - j\dfrac{1}{\omega C_1}} = \frac{20e^{j\,0^\circ}}{10{,}224e^{-j\,38{,}5^\circ}} = \underline{1{,}956e^{j\,38{,}5^\circ}\,A} = \underline{(1{,}531 + j\,1{,}218)\,A}$$

$$\underline{U}_{R_2} = R_2\underline{I}_2 = 8\cdot 1{,}956e^{j\,38{,}5^\circ} = \underline{15{,}648e^{j\,38{,}5^\circ}\,V} = \underline{(12{,}246 + j\,9{,}74)\,V}$$

$$\underline{U}_{C_1} = \frac{\underline{I}_2}{j\omega C_1} = \frac{1{,}956e^{j\,38{,}5^\circ}}{0{,}157e^{j\,90{,}0^\circ}} = \underline{12{,}452e^{-j\,51{,}5^\circ}\,V} = \underline{(7{,}75 - j\,9{,}745)\,V}$$

$$\underline{I} = \underline{I}_1 + \underline{I}_2 = 1{,}030 - j\,2{,}022 + 1{,}531 + j\,1{,}218 = \underline{(2{,}561 - j\,0{,}804)\,A} = \underline{2{,}684e^{-j\,17{,}4^\circ}\,A}$$

$$\underline{U}_{C_2} = \frac{\underline{I}}{j\omega C_2} = \frac{2{,}684e^{-j\,17{,}4^\circ}}{0.314e^{j\,90{,}0^\circ}} = \underline{8{,}543e^{-j\,107{,}4^\circ}\,V} = \underline{(-2{,}555 - j\,8{,}152)\,V}$$

$$\underline{U}_{L_2} = j\omega L_2\underline{I} = 15{,}71e^{j\,90{,}0^\circ}\cdot 2{,}684e^{-j\,17{,}4^\circ} = \underline{42{,}17e^{j\,72{,}6^\circ}\,V} = \underline{(12{,}61 + j\,40{,}24)\,V}$$

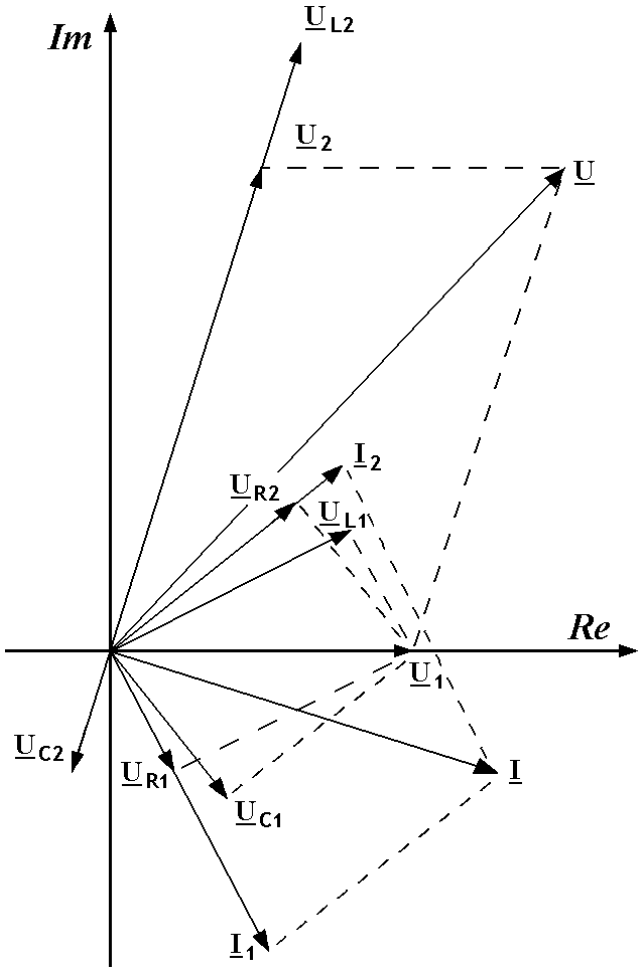
$$\underline{U}_2 = \underline{U}_{C_2} + \underline{U}_{L_2} = -2{,}555 - j\,8{,}152 + 12{,}61 + j\,40{,}24 = \underline{(10{,}06 + j\,32{,}09)\,V} = \underline{33{,}63e^{j\,72{,}6^\circ}\,V}$$

$$\underline{U} = \underline{U}_1 + \underline{U}_2 = 20 + 10{,}06 + j\,32{,}09 = \underline{(30{,}06 + j\,32{,}09)\,V} = \underline{43{,}97e^{j\,46{,}9^\circ}\,V}$$

F. Complex diagram of currents and voltages

Scales :

U : 1 cm ↔ 5 V
I : 1 cm ↔ 0,5 A



EXERCISE IV.3 : CALCULATIONS OF IMPEDANCES

A. Impedances

$$\underline{Z}_1 = R + j\,\omega L + \frac{1}{j\,\omega C} = R + j\,\frac{\omega^2 LC - 1}{\omega C} = \underline{(4 + j\,3{,}04)\ \Omega} = \underline{5e^{j\,37{,}2^\circ}\ \Omega}$$

$$\underline{Z}_2 = \frac{1}{j\,\omega C} + \frac{R \cdot j\,\omega L}{R + j\,\omega L} = \frac{\omega^3 RL^2C - j\,(R^2 + \omega^2 L^2 - \omega^2 R^2 LC)}{\omega C(R^2 + \omega^2 L^2)} = \underline{(3{,}20 - j\,3{,}38)\ \Omega} = \underline{4.7e^{-j\,46{,}5^\circ}\ \Omega}$$

$$\underline{Z}_3 = R + \frac{\dfrac{1}{j\,\omega C} \cdot j\,\omega L}{\dfrac{1}{j\,\omega C} + j\,\omega L} = R + j\,\frac{\omega L}{1 - \omega^2 LC} = \underline{(4 - j\,13{,}12)\ \Omega} = \underline{13.7e^{-j\,73{,}0^\circ}\ \Omega}$$

$$\underline{Z}_4 = j\,\omega L + \frac{R \cdot \dfrac{1}{j\,\omega C}}{R + \dfrac{1}{j\,\omega C}} = \frac{R + j\,\omega(\omega^2 R^2 LC^2 + L - R^2 C)}{1 + (\omega RC)^2} = \underline{(2{,}43 + j\,6{,}06)\ \Omega} = \underline{6.5e^{j\,68{,}2^\circ}\ \Omega}$$

$$\underline{Z}_5 = \frac{\dfrac{R \cdot j\,\omega L}{R + j\,\omega L} \cdot \dfrac{1}{j\omega C}}{\dfrac{R \cdot j\,\omega L}{R + j\,\omega L} + \dfrac{1}{j\,\omega C}} = \frac{\omega^2 R L^2 - j\,\omega R^2 L(\omega^2 LC - 1)}{R^2(\omega^2 LC - 1)^2 + (\omega L)^2} = \underline{(3{,}66 - j\,1{,}12)\ \Omega} = \underline{3.8 e^{-j\,17^\circ}\ \Omega}$$

$$\underline{Z}_6 = \frac{(R + j\,\omega L) \cdot \dfrac{1}{j\,\omega C}}{R + j\,\omega L + \dfrac{1}{j\,\omega C}} = \frac{R - j\,\omega(\omega^2 L^2 C + R^2 C - L)}{(\omega^2 LC - 1)^2 + (\omega RC)^2} = \underline{(3{,}92 - j\,7{,}95)\ \Omega} = \underline{8.6 e^{-j\,63{,}7^\circ}\ \Omega}$$

$$\underline{Z}_7 = \frac{R \cdot \left(j\,\omega L + \dfrac{1}{j\,\omega C}\right)}{R + \left(j\,\omega L + \dfrac{1}{j\,\omega C}\right)} = \frac{R(\omega^2 LC - 1)^2 - j\,\omega R^2 C(\omega^2 LC - 1)}{(\omega^2 LC - 1)^2 + (\omega RC)^2} = \underline{(1{,}46 + j\,1{,}93)\ \Omega} = \underline{2.5 e^{j\,52{,}8^\circ}\ \Omega}$$

B. Currents and voltages (I designates the total current)

Units : V et A. Scales : 1 cm ↔ 1 V ; 1 cm ↔ 0,25 A

1

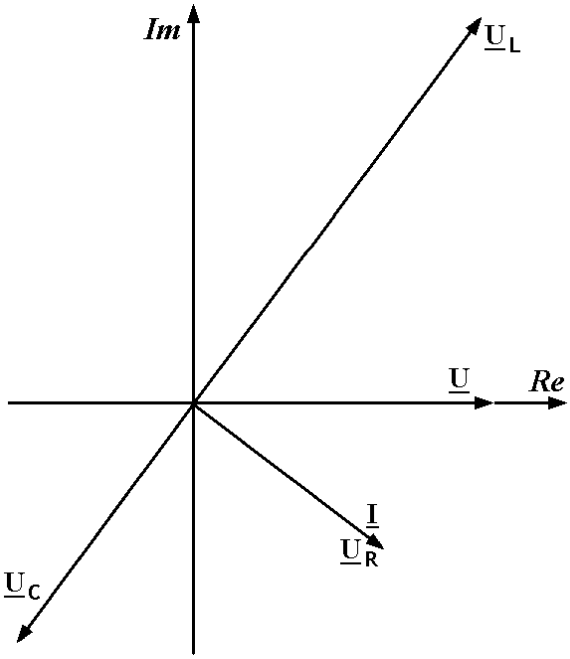
$$\underline{I} = \frac{\underline{U}}{\underline{Z}_1} = \underline{0{,}8 \cdot e^{-j\,37{,}2^\circ}} = \underline{0{,}637 - j\,0{,}484}$$

$$\underline{U}_R = R \cdot \underline{I} = \underline{3{,}2 \cdot e^{-j\,37{,}2^\circ}} = \underline{2{,}549 - j\,1{,}935}$$

$$\underline{U}_L = \underline{Z}_L \cdot \underline{I} = \underline{6{,}409 \cdot e^{j\,52{,}8^\circ}} = \underline{3{,}875 + j\,5{,}105}$$

$$\underline{U}_C = \underline{Z}_C \cdot \underline{I} = \underline{-3{,}979 \cdot e^{-j\,127{,}2^\circ}} = \underline{-2{,}406 - j\,3{,}169}$$

$$\underline{I}_R = \underline{I}_C = \underline{I}_L = \underline{I}$$



2

$$\underline{I} = \frac{\underline{U}}{\underline{Z}_2} = \underline{0{,}85 \cdot e^{j\,46{,}5°}} = \underline{0{,}585 + j\,0{,}617}$$

$$\underline{U}_C = \underline{Z}_c \cdot \underline{I} = \underline{4{,}233 \cdot e^{-j\,43{,}5°}} = \underline{3{,}070 - j\,2{,}914}$$

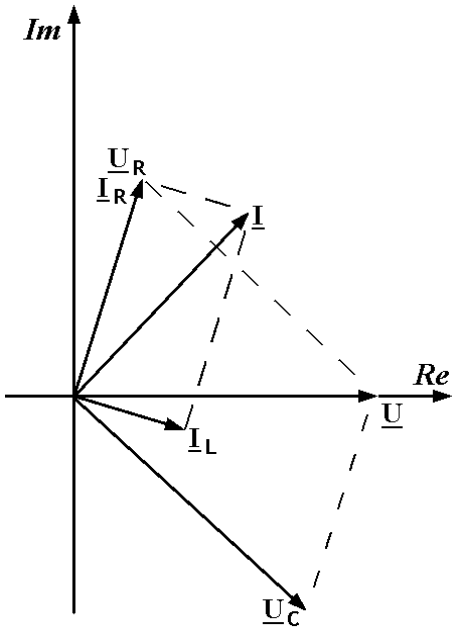
$$\underline{U}_R = \underline{U} - \underline{U}_c = \underline{3{,}058 \cdot e^{j\,72{,}3°}} = \underline{0{,}930 + j\,2{,}914}$$

$$\underline{U}_L = \underline{U}_R$$

$$\underline{I}_R = \frac{\underline{U}_R}{R} = \underline{0{,}764 \cdot e^{j\,72{,}3°}} = \underline{0{,}232 + j\,0{,}729}$$

$$\underline{I}_L = \underline{I} - \underline{I}_R = \underline{1{,}391 \cdot e^{-j\,17{,}6°}} = \underline{0{,}353 - j\,0{,}112}$$

$$\underline{I}_C = \underline{I}$$



3

$$\underline{I} = \frac{\underline{U}}{\underline{Z}_3} = \underline{0{,}292 \cdot e^{j\,73°}} = \underline{0{,}085 + j\,0{,}279}$$

$$\underline{U}_R = R \cdot \underline{I} = \underline{1{,}168 \cdot e^{j\,73°}} = \underline{0{,}341 + j\,1{,}116}$$

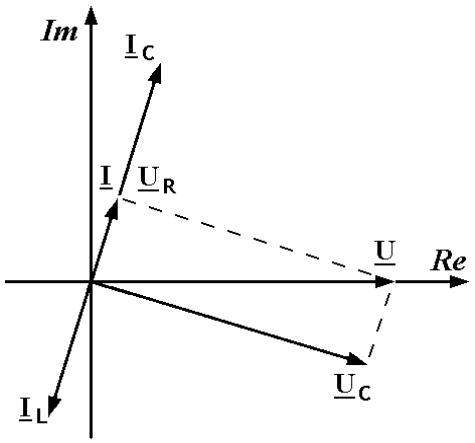
$$\underline{U}_C = \underline{U} - \underline{U}_R = \underline{3{,}826 \cdot e^{-j\,17°}} = \underline{3{,}66 - j\,1{,}116}$$

$$\underline{U}_L = \underline{U}_C$$

$$\underline{I}_C = \frac{\underline{U}_C}{\underline{Z}_C} = \underline{0{,}769 \cdot e^{j\,73°}} = \underline{0{,}224 + j\,0{,}736}$$

$$\underline{I}_L = \underline{I} - \underline{I}_C = \underline{0{,}478 \cdot e^{-j\,107°}} = \underline{-0{,}139 - j\,0{,}457}$$

$$\underline{I}_R = \underline{I}$$



4

$$\underline{I} = \frac{\underline{U}}{\underline{Z}_4} = \underline{0{,}615 \cdot e^{j\,-68{,}2°}} = \underline{0{,}2285 - j\,0{,}5714}$$

$$\underline{U}_L = \underline{Z}_L \cdot \underline{I} = \underline{4{,}93 \cdot e^{j\,21{,}8°}} = \underline{4{,}577 + j\,1{,}831}$$

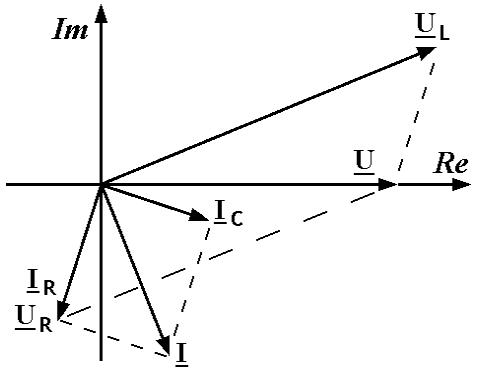
$$\underline{U}_R = \underline{U} - \underline{U}_L = \underline{1{,}92 \cdot e^{-j\,107{,}5°}} = \underline{-0{,}5773 - j\,1{,}831}$$

$$\underline{U}_C = \underline{U}_R$$

$$\underline{I}_R = \frac{\underline{U}_R}{R} = \underline{0{,}48 \cdot e^{-j\,107{,}5°}} = \underline{-0{,}144 - j\,0{,}458}$$

$$\underline{I}_C = \underline{I} - \underline{I}_R = \underline{0{,}386 \cdot e^{-j\,17{,}5°}} = \underline{0{,}368 - j\,0{,}116}$$

$$\underline{I}_L = \underline{I}$$



5

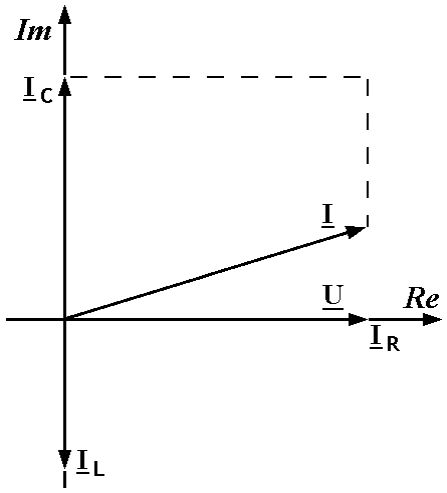
$$\underline{I} = \frac{\underline{U}}{\underline{Z}_5} = \underline{1{,}045 \cdot e^{j\,16{,}9°}} = \underline{1 + j\,0{,}304}$$

$$\underline{U}_R = \underline{U}_L = \underline{U}_C = \underline{U}$$

$$\underline{I}_R = \frac{\underline{U}}{R} = \underline{1 \cdot e^{-j\,0°}} = \underline{1 + j\,0}$$

$$\underline{I}_L = \frac{\underline{U}}{\underline{Z}_L} = \underline{0{,}5 \cdot e^{-j\,90°}} = \underline{0 - j\,0{,}5}$$

$$\underline{I}_C = \underline{I} - \underline{I}_R - \underline{I}_L = \underline{0{,}804 \cdot e^{j\,90°}} = \underline{0 + j\,0{,}804}$$



6

$$\underline{I} = \frac{\underline{U}}{\underline{Z}_6} = \underline{0{,}465 \cdot e^{j\,63{,}7°}} = \underline{0{,}206 + j\,0{,}418}$$

$$\underline{I}_C = \frac{\underline{U}}{\underline{Z}_C} = \underline{0{,}804 \cdot e^{j\,90°}} = \underline{0 + j\,0{,}804}$$

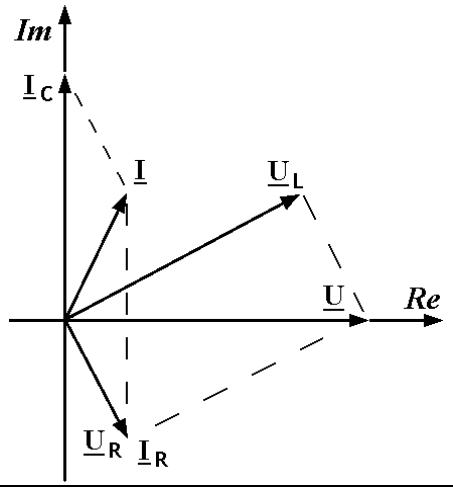
$$\underline{I}_R = \underline{I} - \underline{I}_C = \underline{0{,}439 \cdot e^{-j\,62°}} = \underline{0{,}206 - j\,0{,}387}$$

$$\underline{I}_L = \underline{I}_R$$

$$\underline{U}_R = R \cdot \underline{I}_R = \underline{1{,}754 \cdot e^{-j\,62°}} = \underline{0{,}824 - j\,1{,}548}$$

$$\underline{U}_L = \underline{U} - \underline{U}_R = \underline{3{,}514 \cdot e^{j\,28°}} = \underline{3{,}103 + j\,1{,}651}$$

$$\underline{U}_C = \underline{U}$$



7

$$\underline{I} = \frac{\underline{U}}{\underline{Z}_7} = \underline{1{,}654 \cdot e^{-j\,52{,}8°}} = \underline{1 - j\,1{,}317}$$

$$\underline{I}_R = \frac{\underline{U}}{R} = \underline{1 \cdot e^{-j\,0°}} = \underline{1 + j\,0}$$

$$\underline{I}_L = \underline{I} - \underline{I}_R = \underline{1{,}317 \cdot e^{-j\,90°}} = \underline{0 - j\,1{,}317}$$

$$\underline{I}_C = \underline{I}_L$$

$$\underline{U}_L = \underline{Z}_L \cdot \underline{I}_L = \underline{10{,}55 \cdot e^{j\,0°}} = \underline{10{,}55 + j\,0}$$

$$\underline{U}_C = \underline{U} - \underline{U}_L = \underline{6{,}55 \cdot e^{j\,180°}} = \underline{-6{,}55 + j\,0}$$

$$\underline{U}_R = \underline{U}$$

Scale : 1 cm ↔ 2 V ; 1 cm ↔ 0,5 A